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DEFORMATION RETRACTS & HOMOTOPY EQUIVALENCES

Def: Let $A \subset X$ be a any. A is a "deformation retract of X" if id_X is homotopic to a continuous map $f: X \to A$ s.t. $f|_A = id_A$.

The homotopy is called a "deformation retraction of X onto A", and f a "retraction of X onto A".

Theorem: Let A be a deformation retract of X. Then the $j: (A, x_0) \hookrightarrow (X, x_0)$ induces an isomorphism of fundamental groups.

Proof: WTS: j_* is an isomorphism.

$\exists\, H: [0,1] \times X \longrightarrow X$

s.t. $H_0 = id_X$, $H_1 = f: X \longrightarrow A$

Claim: $j_* \circ f_* = id: \pi_1(X, x_0)$ ✓

$f_* \circ j_* = id: \pi_1(A, x_0)$ ✓

Claim 1 $\Rightarrow$ Prop.

Proof of claim 1: Note $f: X \longrightarrow A$ is s.t. $f|_A = id_A$.

$\Rightarrow f_* \circ j_* = id$ on $\pi_1(A, x_0)$.

Prop: If $f: X \longrightarrow Y$ and $g: X \longrightarrow Y$ are homotopic and $f(x_0) = g(x_0) = y_0$. Then $f_* = g_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

Proof of prop: Let H be a homotopy between f and g.

ie, $H_0 = f$ and $H_1 = g$.

Then note: $\forall$ loop $[\gamma] \in \pi_1(X, x_0)$, $H_t(\gamma)$ gives an isotopy between $H_0(\gamma) = f(\gamma)$ and $H_1(\gamma) = g(\gamma)$.

$\Rightarrow [f \circ \gamma] = [g \circ \gamma] \Rightarrow f_*([\gamma]) = g_*([\gamma])$

Prop $\Rightarrow f_* = id_*$, which would prove claim 1 and the original prop.

Example:

1. $\mathbb{R}^3 \setminus z$ axis deformation retracts onto xy plane $\setminus 0$ in $\mathbb{R}^3$.

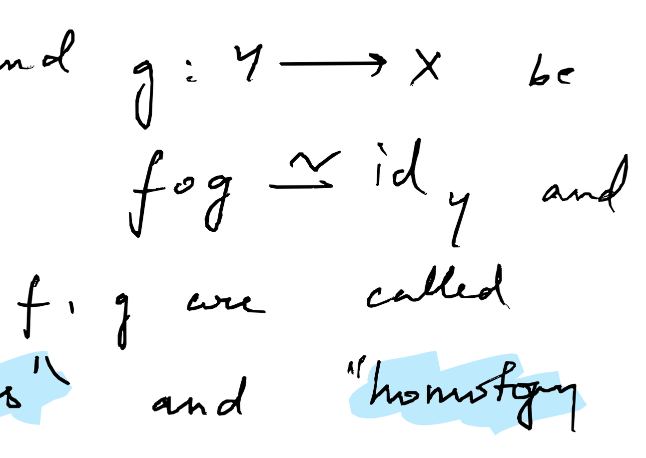
Prop: $H_t(x, y, z) = (x, y, (1-t)z)$.

H is a retract from $\mathbb{R}^3$ to $\mathbb{R}^2$. & restricts to one from $\mathbb{R}^3 \setminus z$ axis to $\mathbb{R}^2 \setminus \{0\}$

Corollary: $\pi_1(\mathbb{R}^3 \setminus z \text{ axis}, x_0) = \mathbb{Z}$

2. Doubly punctured plane has figure 8 as its deformation retract.

$\mathbb{R}^2 \setminus \{(-1,0), (1,0)\}$,



Proof: Any point in $\mathbb{R}^2 \setminus \{x_0, x_1\}$ lies on a "thickened" figure 8.

Def: Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ be cts maps s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. Then f, g are called "homotopy equivalences" and "homotopy inverses" of each other.

Def: Let X be a topological space and $x_0, x_1 \in X$ given a path α in X from x_0 to x_1, we define

$$\hat{\alpha}: \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

as $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$

Theorem: $\hat{\alpha}: \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$ is a group isomorphism.

Prop: If X is path connected and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Lemma: Let $h, k: X \longrightarrow Y$ be cts,

$h(x_0) = y_0$
$k(x_0) = y_1$.

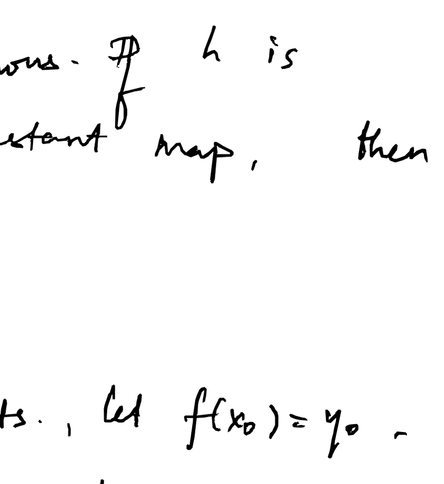
If h, k are homotopic, there is a path α in Y from y_0 to y_1 s.t. $k_* = \hat{\alpha} \circ h_*$. Indeed, if $H: [0,1] \times X \longrightarrow Y$ is the homotopy between h and k, then α is the path $\alpha(t) = H(t, x_0)$.

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "h_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\hat{\alpha}"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

Proof: recall $k_*([f]) = [k(f)]$ and $h_*([f]) = [h(f)]$.

$\hat{\alpha} \circ h_*([f])$ is the loop $[\bar{\alpha} * h(f) * \alpha]$, which starts and ends at y_1.

Note that for any t, $H_t(h(f))$ is homotopic to $\overline{\alpha|_{[0,t]}} * h(f) * \alpha|_{[0,t]}$



Since $H_1(h(f)) = k(f)$, the result follows.

Corollary: If h_* is one-to-one (or) onto (or) constant, so is k_*.

Corollary: Let $h: X \longrightarrow Y$ be continuous. If h is homotopy equivalent to a constant map, then h_* is a constant map.

Theorem. Let $f: X \longrightarrow Y$ be cts., let $f(x_0) = y_0$. If f is a homotopy equivalence, then $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is an isomorphism.

Proof: Let g be the homotopy inverse of f.

Note: $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

$\therefore (f \circ g)_* = \hat{\alpha} \circ (id_Y)_*$ and $(g \circ f)_* = \hat{\beta} \circ (id_X)_*$

for paths α, β in Y, X resp.

since $\hat{\alpha}, (id_Y)_*, \hat{\beta}, (id_X)_*$ are isomorphisms, so are $(f \circ g)_* = f_* \circ g_*$ and $(g \circ f)_* = g_* \circ f_*$. This shows f_*, g_* are isomorphisms.